Carl von Ossietzky Universität Oldenburg

Towards a Formalisation of Justification and Justifiability

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Motivation

We are in the process of building complex autonomous systems that

- perceive their environment,
- exchange information, and
- decide independently on their actions based on their understanding of the world.

Opportunity: The variety of perspectives leads to more complete worldview.

Problem: Lots of information from different sources can lead to conflicting worldview.

Requirement: A unified formal method that

- express individual and collective knowledge and beliefs,
- analyse and justify individual or collective decisions or conflicts.

→ **a formal epistemology for autonomous systems**

https://www.ich-bin-zukunft.de/allgemein/autonome-autos/

Belief, Knowledge, and Justification — a Long Story Short

369 BC Plato: **knowledge is justified true belief.** (JTB)

Since 1950 Formalisation of knowledge and belief in **epistemic** modal logics:

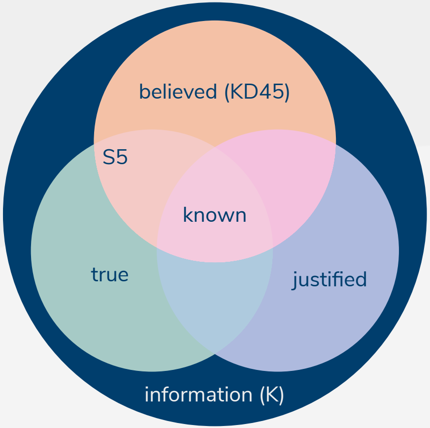
K Basic normal modal logic of 'information'. Information is closed under logical consequence.

KD45 Belief is **consistent**. $(\Box\bot \to \bot)$.
Introspection principles hold:
To believe is to believe to believe.
Not to believe is to believe not to believe.

S5 Knowledge is **true belief.** $(\Box\phi \to \phi)$
Whenever ϕ is known, then ϕ must also be true.

After 1950 Various formal methods dealing with knowledge, belief, and its representation or change: description logic, AGM belief revision, temporal logics, dynamic epistemic logics, BDI, ...

Since 2006 Artemov's Justification Logic **J** supplies the missing third component of knowledge as justified true belief [1].



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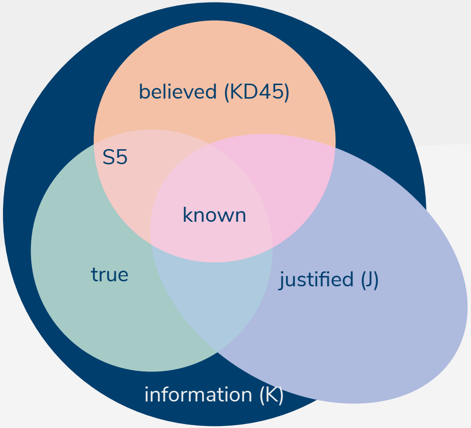

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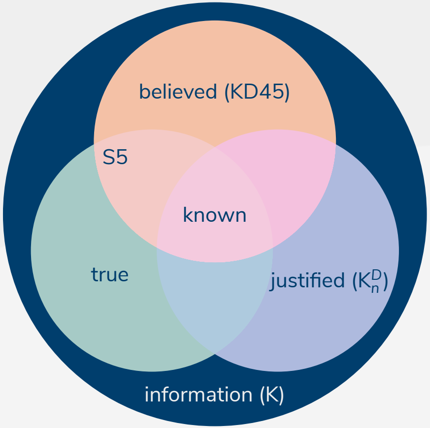
Today Cut Artemov's justification logic back to normal modal logics leads to well-known logic K_n^D. [3]



Knowledge is justified true belief (JTB)

From Artemov's Justification Logic to K_n^D

Artemov's Justification Logic *J* [1]

- Justification terms are abstractions of a logical proof.
- **Principle of justification:**

$$s{:}(\phi \to \psi) \to t{:}\phi \to (s \cdot t){:}\psi$$

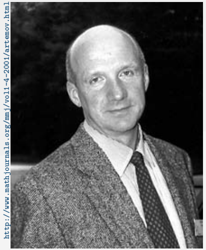
- Example:

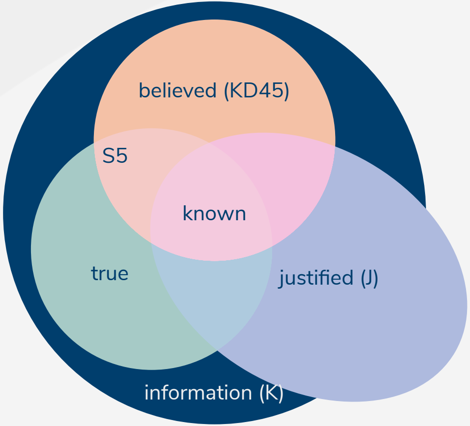
$$\frac{s{:}(\mathtt{rain} \to \mathtt{wet}), \quad t{:}\mathtt{rain}}{(s \cdot t){:}\mathtt{wet}}$$

- Propositional tautologies, like $A \lor \neg A$, are not justified ex officio.
- Yields rather complex justification terms, like $((s \cdot (s \cdot t)) \cdot (t \cdot s))$.

Sergei Nikolaevich Artemov (1951–) is a logician. He is best known for his invention of logics of proofs and justifications.

http://www.mathjournals.org/mmj/vol1-4-2001/artemov.html





From Artemov's Justification Logic to K_n^D

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Today Let $\cdot$ be set union,
use modal $\Box$, $\Diamond$-notation with $\Diamond\phi \equiv \neg\Box\neg\phi$

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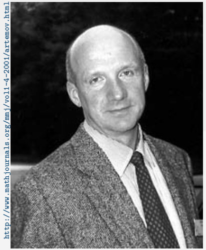

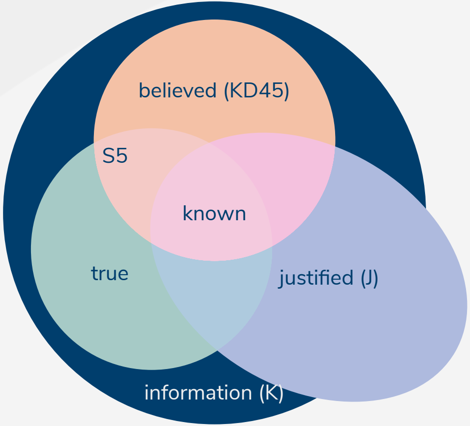

From Artemov's Justification Logic to K_n^D

Logic with distributed information K_n^D [3]

- Justification terms are abstractions of a logical proof.
- **Principle of justification / information distribution:**

  $$\Box_s(\phi \to \psi) \to \Box_t\phi \to \Box_{s\cdot t}\psi$$

- Example:

  $$\frac{\Box_s(\texttt{rain} \to \texttt{wet}), \quad \Box_t\texttt{rain}}{\Box_{s\cdot t}\texttt{wet}}$$

- Propositional tautologies, like $A \vee \neg A$, are ~~not~~ justified ex officio.
- Yields simple justification terms, like $\Box_{s\cdot t}$.

Today Let $\cdot$ be set union,
use modal $\Box$, $\Diamond$-notation with $\Diamond\phi \equiv \neg\Box\neg\phi$

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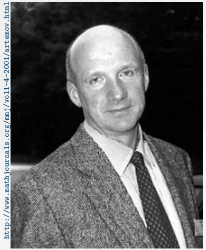

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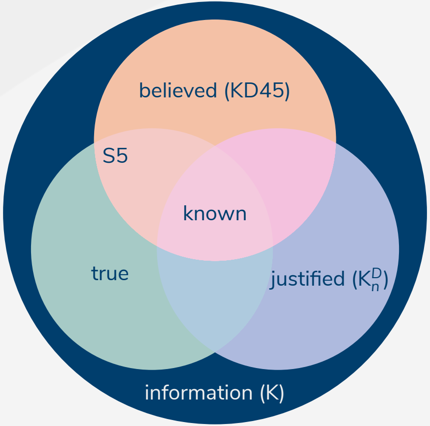
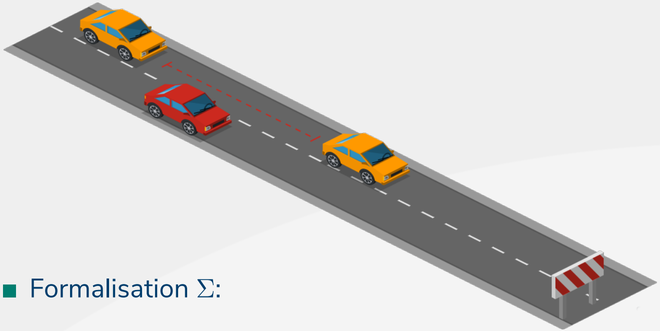


Example

- A vehicle with an automated driving system (ADS) is driving on a multi-lane road. As its lane is blocked, it has to decide whether it should brake or use the small gap for the lane change.
- Goals:
  - (g_1) Drive fast (as the user is in a hurry).
  - (g_2) Considerate driving behaviour (as default).
- Observations:
  - (o_1) The current lane is blocked.
  - (o_2) There is a gap on the neighbouring lane — just large enough for a lane change.
- Rules:
  - (r_1) Whenever the lane is blocked, brake or change lane.
  - (r_2) Never change lane when no gap is available.
  - (r_3) Never brake when the goal is to drive fast.
  - (r_4) When the goal is to drive considerately, then change lane only if there is a large gap.
- The totality of given information is inconsistent!



- Formalisation Σ:

$$\Box_{g_1}(\mathtt{beFast})$$
$$\Box_{g_2}(\mathtt{beConsiderate})$$
$$\Box_{o_1}(\mathtt{laneBlocked}),$$
$$\Box_{o_2}(\mathtt{gapAvail} \wedge \neg\mathtt{largeGapAvail})$$
$$\Box_{r_1}(\mathtt{laneBlocked} \to \mathtt{brake} \vee \mathtt{changeLane})$$
$$\Box_{r_2}(\mathtt{changeLane} \to \mathtt{gapAvail})$$
$$\Box_{r_3}(\mathtt{beFast} \to \neg\mathtt{brake})$$
$$\Box_{r_4}(\mathtt{beConsiderate} \wedge \mathtt{changeLane} \to \mathtt{largeGapAvail})$$

Justifications

- We prove the inconsistency using K_n^D with the following **principle of justification:** $\Box_s(\phi \to \psi) \to \Box_t\phi \to \Box_{s\cdot t}\psi$
- The proof:

$$r_3, g_1 \vdash \Box_{r_3 \cdot g_1} \neg\mathtt{brake} \quad (1)$$

$$o_1, r_1 \vdash \Box_{o_1 \cdot r_1}(\mathtt{brake} \lor \mathtt{changeLane}) \quad (2)$$

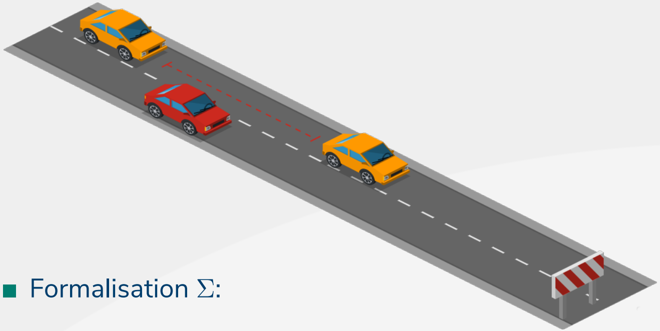
$$(1),(2) \vdash \Box_{o_1 \cdot r_1 \cdot r_3 \cdot g_1} \mathtt{changeLane} \quad (3)$$

$$r_4, g_2 \vdash \Box_{r_4 \cdot g_2}(\mathtt{changeLane} \to \mathtt{largeGapAvail}) \quad (4)$$

$$(3),(4) \vdash \Box_{o_1 \cdot r_1 \cdot r_3 \cdot r_4 \cdot g_1 \cdot g_2} \mathtt{largeGapAvail} \quad (5)$$

$$o_2, (5) \vdash \Box_{o_1 \cdot o_2 \cdot r_1 \cdot r_3 \cdot r_4 \cdot g_1 \cdot g_2} \bot \quad (6)$$

- $\Box_{o_1 \cdot o_2 \cdot r_1 \cdot r_3 \cdot r_4 \cdot g_1 \cdot g_2}$ is a **justification** for $\bot$. It states which observations, rules, and goals have been used to derive the contradiction.



- Formalisation Σ:

$\Box_{g_1}(\mathtt{beFast})$

$\Box_{g_2}(\mathtt{beConsiderate})$

$\Box_{o_1}(\mathtt{laneBlocked})$,

$\Box_{o_2}(\mathtt{gapAvail} \land \neg\mathtt{largeGapAvail})$

$\Box_{r_1}(\mathtt{laneBlocked} \to \mathtt{brake} \lor \mathtt{changeLane})$

$\Box_{r_2}(\mathtt{changeLane} \to \mathtt{gapAvail})$

$\Box_{r_3}(\mathtt{beFast} \to \neg\mathtt{brake})$

$\Box_{r_4}(\mathtt{beConsiderate} \land \mathtt{changeLane} \to \mathtt{largeGapAvail})$

Justifiability

Atoms are the information sources
$\mathcal{S} = \{r_1, r_2, r_3, r_4, g_1, g_2, o_1, o_2\}$
Instances are nonempty subsets of $\mathcal{S}$

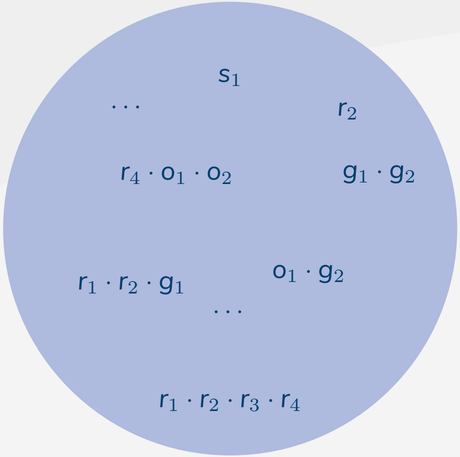
- **What is justifiability?**
  - ϕ is justifiable if and only if there **exists** a consistent instance s of $\mathcal{S}$ that justifies ϕ.
- **Why justifiability?**
  - ϕ justifiable $\Rightarrow$ good reasons for ϕ,
    $\neg\phi$ not justifiable $\Rightarrow$ no evidence against ϕ,
    even if $\mathcal{S}$ is inconsistent
- **Formally**
  - Add quantifiers $\Rightarrow$ $QK^{D}_{\mathcal{S}}$:
    ϕ is justifiable if and only if $\exists_{x \subseteq \mathcal{S}}(\Diamond_x \top \wedge \Box_x \phi)$.

(Quantification for justification proposed by Fitting in [4].
Supposed to be undecidable.)

$\exists$

Quantify over instances:

$\forall$



Axiomatisation

Definition ($K^D_\mathcal{S}$ and simplified $QK^D_\mathcal{S}$)

Let $\mathcal{S}$ be a **finite** set of atoms and $\mathcal{V}$ be an enumerable set of propositional variables. The logic $K^D_\mathcal{S}$ / $QK^D_\mathcal{S}$ is given by the rules and axioms for all instances s and t of $\mathcal{S}$

$K_\mathcal{S}$	from $\vdash \phi$ and $\vdash \phi \to \psi$ infer $\vdash \psi$;	(MP)
	$\vdash \phi$ if ϕ is an instance of a prop. tautology;	(0)
	from $\vdash \phi$ infer $\vdash \Box_s \phi$;	(Nec)
	$\vdash \Box_s(\phi \to \psi) \to \Box_s\phi \to \Box_s\psi$;	(K)
$K^D_\mathcal{S} = K_\mathcal{S}$ plus	$\vdash \Box_s\phi \to \Box_t\phi$ if $s \subseteq t$.	(Dist)
$QK^D_\mathcal{S} = K^D_\mathcal{S}$ plus	$\vdash \exists_{x\subseteq t}(\phi) \leftrightarrow \bigvee_{r\subseteq t} \phi[r/x]$	(Ex)
	$\vdash \forall_{x\subseteq t}(\phi) \leftrightarrow \bigwedge_{r\subseteq t} \phi[r/x]$	(Fa)

- 'Justification principle' $\Box_s(\phi \to \psi) \to \Box_t\phi \to \Box_{s\cdot t}\psi$ is a consequence of $K^D_\mathcal{S}$.
- Quantification in $QK^D_\mathcal{S}$ is very restrictive, as there is only one variable symbol.
- (Ex) and (Fa) explicitly enumerate all instances of $\mathcal{S}$!

Semantics

Definition (Kripke structure)

A **multimodal Kripke structure for** $\mathcal{S}$ is the tuple $M = (\Omega, \mapsto, \pi)$, where

1. Ω is a nonempty set of **possible worlds**;
2. $\mapsto_s$ is a binary **accessibility relation** over Ω for any $s \subseteq \mathcal{S}$.
3. $\pi(\omega)$ is the **truth assignment** for $\omega \in \Omega$.

Definition (Semantics of $QK^D_{\mathcal{S}}$)

Multi modal Kripke structure M with an **interpretation of** x given by $\beta \subseteq \mathcal{S}$:

$$
\begin{aligned}
&(M, \omega, \beta) \not\models \bot, \quad (M, \omega, \beta) \models A \iff A \in \pi(\omega),\\
&(M, \omega, \beta) \models \phi \rightarrow \psi \iff (M, \omega, \beta) \models \phi \text{ implies } (M, \omega, \beta) \models \psi,\\
&(M, \omega, \beta) \models \Box_s \phi \iff (M, \omega', \beta) \models \phi \text{ for all } \omega' \in \Omega \text{ with } \omega \mapsto_s \omega',\\
&(M, \omega, \beta) \models \Box_x \phi \iff (M, \omega', \beta) \models \phi \text{ for all } \omega' \in \Omega \text{ with } \omega \mapsto_\beta \omega',\\
&(M, \omega, \beta) \models \forall_{x \subseteq t} \phi \iff (M, \omega, \beta') \models \phi \text{ for all } \beta' \in \mathcal{S} \text{ with } \beta' \subseteq t
\end{aligned}
$$

We write $(M, \omega) \models \phi$ if there exists an interpretation β with $(M, \omega, \beta) \models \phi$.

- $QK^D_{\mathcal{S}}$ is a trivial extension of $K^D_{\mathcal{S}}$.
- Quantification can also be restricted by a lower bound.
- Immediately, it follows that $QK^D_{\mathcal{S}}$ is sound, complete and decidable.
- Decidability suffers from exponential blowup due to explicit enumeration!

Automatic Reasoning in QK_S^D

Prototypical satisfiability solver **episat** for QK_S^D

- tableau based
- supports lower and upper bounds for $\exists_x$ and $\forall_x$
- tableau unrolled into a Boolean satisfiability problem (similar to Inkresat [5])
- bit vector encoding of instances
  - for interpretations
  - for “owners” of possible worlds
- avoids explicit enumeration
- `https://vhome.offis.de/~willemh/episat/`

```
begin
# Hierarchy
  r > r1,r2,r3,r4;
  o > o1,o2,r;
  g > g1,g2,o;
  s > g;

  # r inherits information from r1, r2, r3, r4
  # o inherits from o1, o2 and r
  # g inherits from g1, g2 and o
  # s inherits from g

  # Sigma
  [o1] laneBlocked;
  [o2]( gapAvailable & ~largeGapAvailable );
  [r1]( laneBlocked -> brake | changeLane );
  [r2]( changeLane -> gapAvailable );
  [r3]( beFast -> ~brake );
  [r4]( beConsiderate & changeLane -> largeGapAvailable );
  [g1]( beFast );

  # We replace the goal g2
  #    [g2]( beConsiderate );
  # by the default rule
  ( Fa x, x <= o,g1,g2. ( <x>true -> <x>beConsiderate ) -> [g] beConsiderate );
  # I.e., whenever all consistent subinstances of {o,g1,g2}
  # consider beConsiderate as possible, then beConsiderate will be necessary for
  # the superinstance g.

  # Is the system now consistent?
  Fa x, x <= s. ( <x> true );
end
```

example.episat

```
$ ./episat example.episat
satisfiable!
[...]
```

Tableau for QK^D_S

- Classical rules for $\wedge$ and $\vee$:

$$(\wedge)\ \frac{(\omega,\beta)\models\phi\wedge\psi}{\begin{array}{c}(\omega,\beta)\models\phi\\(\omega,\beta)\models\psi\end{array}}\qquad(\vee)\ \frac{(\omega,\beta)\models\phi\vee\psi}{(\omega,\beta)\models\phi\mid(\omega,\beta)\models\psi}$$

- $\Diamond_\mathsf{s}$- and $\Box_\mathsf{s}$-rules from [5] and [2]:

$$(\Diamond_\mathsf{s})\ \frac{(\omega,\beta)\models\Diamond_\mathsf{s}\phi}{\begin{array}{c}(\omega',\beta)\models(\mathsf{s}\subseteq\mathbf{p})\wedge\phi\\\omega\mapsto\omega',\quad\omega'\ \text{new}\end{array}}\qquad(\Box_\mathsf{s})\ \frac{(\omega,\beta)\models\Box_\mathsf{s}\phi,\quad\omega\mapsto\omega'}{(\omega',\beta)\models(\mathsf{s}\subseteq\mathbf{p})\to\phi}$$

- $\Diamond_\mathsf{x}$- and $\Box_\mathsf{x}$-rules for variable modal operators depending on the assignment β for **x**:

$$(\Diamond_\mathsf{x})\ \frac{(\omega,\beta)\models\Diamond_\mathsf{x}\phi}{\begin{array}{c}(\omega',\beta)\models(\mathbf{x}\subseteq\mathbf{p})\wedge\phi\\\omega\mapsto\omega',\quad\omega'\ \text{new}\end{array}}\qquad(\Box_\mathsf{x})\ \frac{(\omega,\beta)\models\Box_\mathsf{x}\phi,\quad\omega\mapsto\omega'}{(\omega',\beta)\models(\mathbf{x}\subseteq\mathbf{p})\to\phi}$$

- $\exists$-rule introduces a new valuation β' for **x** constrained by $\mathbf{x}\subseteq\mathsf{s}$. The $\forall_0$-rule generates a substitution instance for the upper bound.

$$(\exists)\ \frac{(\omega,\beta)\models\exists_{\mathsf{x}\subseteq\mathsf{s}}(\phi)}{\begin{array}{c}(\omega,\beta')\models(\mathbf{x}\subseteq\mathsf{s})\wedge\phi\\\beta'\ \text{new}\end{array}}\qquad(\forall_0)\ \frac{(\omega,\beta)\models\forall_{\mathsf{x}\subseteq\mathsf{s}}(\phi)}{(\omega,\beta)\models\phi[\mathsf{s}/\mathsf{x}]}$$

- $\forall_\mathsf{s}$- and $\forall_\mathsf{x}$-rules generates substitution instance of ϕ on demand

$$(\forall_\mathsf{s})\ \frac{(\omega,\beta)\models\forall_{\mathsf{x}\subseteq\mathsf{s}}(\phi),\quad(\omega,\beta')\models\Diamond_\mathsf{t}\psi\ \text{or}\ (\omega,\beta')\models\Box_\mathsf{t}\psi}{(\omega,\beta')\models(\mathsf{t}\subseteq\mathsf{s})\to\phi[\mathsf{t}/\mathsf{x}]}$$

$$(\forall_\mathsf{x})\ \frac{(\omega,\beta)\models\forall_{\mathsf{x}\subseteq\mathsf{s}}(\phi),\quad(\omega,\beta')\models\Diamond_\mathsf{x}\psi\ \text{or}\ (\omega,\beta')\models\Box_\mathsf{x}\psi}{(\omega,\beta')\models(\mathbf{x}\subseteq\mathsf{s})\to\phi}$$

p for “owners”, **x** for “interpretations”

Conclusion and Outlook

Conclusion

- QK_S^D is a normal logic that supports quantification over modalities.
- tableau solver avoids explicit enumeration
- justifications similar to Artemov's justification logic.
- embedded in the family of normal modal logics
- a logical framework for dealing with inconsistent information — without discarding information

https://www.dreamstime.com/

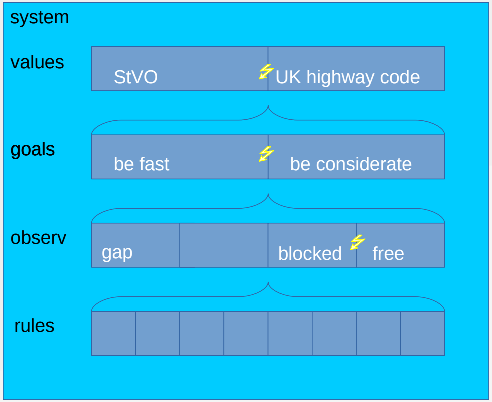
https://www.pxfuel.com/

Outlook / Future Work

Layered systems

- Idea: `beConsiderate` if there is no evidence against it.
- $\forall_{x \subsetneq g}(\Diamond_x \top \rightarrow \Diamond_x \texttt{beConsiderate})) \rightarrow \Box_g \texttt{beConsiderate}$
- Additional $\subseteq$-constraints on $\mathcal{S}$
  - $\cdot$ for horizontal layers — the dimension of justification
  - $\subseteq$ for vertical layers — the dimension of causal relations
  - already supported by the solver
  - preliminary paper available at
    https://vhome.offis.de/~willemh/episat/

- Adding positive and negative introspection principles
- Comparison with other solvers, SPASS is the canonical candidate
- Temporal, first-order, probabilistic(?) extensions...
- Practical applications and case studies



https://culture.kissflow.com/be-an-addict-of-introspection-9e01d50be90/

References I

[1] Sergei N. Artemov and Melvin Fitting. “Justification Logic”. In: **The Stanford Encyclopedia of Philosophy**. Ed. by Edward N. Zalta. Summer 2020. Metaphysics Research Lab, Stanford University, 2020.

[2] Patrick Blackburn, Johan van Benthem, and Frank Wolter. **Handbook of modal logic**. Elsevier, 2006.

[3] Ronald Fagin et al. **Reasoning About Knowledge**. Cambridge, MA, USA: MIT Press, 2003. ISBN: 0262562006.

[4] Melvin Fitting. “A quantified logic of evidence”. In: **Annals of Pure and Applied Logic** 152.1 (2008), pp. 67–83. ISSN: 0168-0072. DOI: `10.1016/j.apal.2007.11.003`.

[5] Mark Kaminski and Tobias Tebbi. “InKreSAT: modal reasoning via incremental reduction to SAT”. In: **Automated Deduction – CADE-24**. Ed. by Maria Paola Bonacina. Springer. 2013, pp. 436–442. DOI: `10.1007/978-3-642-38574-2\_31`.

Example

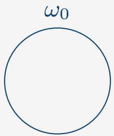
Tableau:

$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \rightarrow B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$

Constraints:

Solver assignments:

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \rightarrow B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



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$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ $(\exists)$

Constraints:

Solver assignments:

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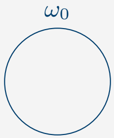
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Color code:
under consideration | worked off | solver

Example

Tableau:

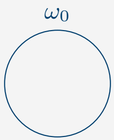
$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$
$(\omega_0, \beta_1) \models \Diamond_x \top$
$(\omega_0, \beta_1) \models \Box_x B$

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$

Solver assignments:

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Color code:
under consideration | worked off | solver

Example

Tableau:

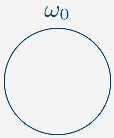
$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \rightarrow B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ (?)
$(\omega_0, \beta_1) \models \Diamond_x \top$
$(\omega_0, \beta_1) \models \Box_x B$

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$

Solver assignments:

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \rightarrow B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Color code:
under consideration | worked off | solver

Example

Tableau:

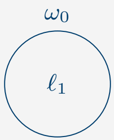
$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \land \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$
$(\omega_0, \beta_1) \models \Box_x B$

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$

Solver assignments:
$\mathbf{x}(\beta_1) := s$

Model for:
$\Box_r \neg B \land \Box_s A \land \Box_t (A \to B) \land \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \land \Box_x B)$



Color code:
under consideration | worked off | solver

Example

Tableau:

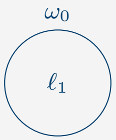
$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ $(\Diamond_x)$
$(\omega_0, \beta_1) \models \Box_x B$

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$

Solver assignments:
$\mathbf{x}(\beta_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Color code:
under consideration | worked off | solver

Example

Tableau:

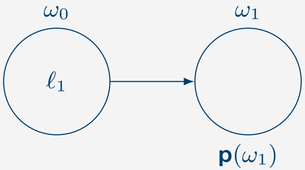
$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓
$(\omega_0, \beta_1) \models \Box_x B$
$(\omega_1, \beta_1) \models \ell_2$
$(\omega_1, \beta_1) \models \top$

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$

Solver assignments:
$\mathbf{x}(\beta_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Color code:
under consideration | worked off | solver

Example

Tableau:

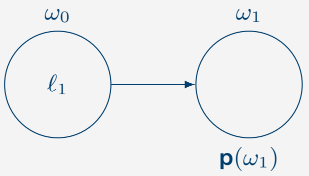
$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓
$(\omega_0, \beta_1) \models \Box_x B$
$(\omega_1, \beta_1) \models \ell_2$ (?)
$(\omega_1, \beta_1) \models \top$ ✓

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$

Solver assignments:
$\mathbf{x}(\beta_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Color code:
under consideration | worked off | solver

Example

Tableau:

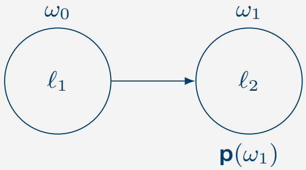
$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓
$(\omega_0, \beta_1) \models \Box_x B$
$(\omega_1, \beta_1) \models \ell_2$ ✓
$(\omega_1, \beta_1) \models \top$ ✓

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$

Solver assignments:
$\mathbf{x}(\beta_1) := s$
$\mathbf{p}(\omega_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Color code:
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Example

Tableau:

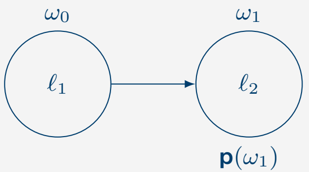
$(\omega_0, \beta_0) \models \Box_r \neg B$ $(\Box_r)$
$(\omega_0, \beta_0) \models \Box_s A$ $(\Box_s)$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$ $(\Box_r)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓ enables $\Box$
$(\omega_0, \beta_1) \models \Box_x B$ $(\Box_x)$
$(\omega_1, \beta_1) \models \ell_2$ ✓
$(\omega_1, \beta_1) \models \top$ ✓

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$

Solver assignments:
$\mathbf{x}(\beta_1) := s$
$\mathbf{p}(\omega_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Color code:
under consideration | worked off | solver

Example

Tableau:

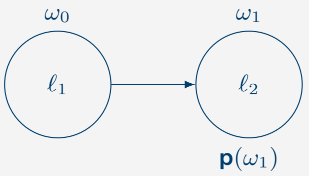
$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓
$(\omega_0, \beta_1) \models \Box_x B$
$(\omega_1, \beta_1) \models \ell_2$ ✓
$(\omega_1, \beta_1) \models \top$ ✓
$(\omega_1, \beta_0) \models \ell_3 \to \neg B$
$(\omega_1, \beta_0) \models \ell_4 \to A$
$(\omega_1, \beta_0) \models \ell_5 \to (A \to B)$
$(\omega_1, \beta_1) \models \ell_6 \to B$

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$
$\ell_3 \iff r \subseteq \mathbf{p}(\omega_1)$
$\ell_4 \iff s \subseteq \mathbf{p}(\omega_1)$
$\ell_5 \iff t \subseteq \mathbf{p}(\omega_1)$
$\ell_6 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$

Solver assignments:
$\mathbf{x}(\beta_1) := s$
$\mathbf{p}(\omega_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Color code:
under consideration | worked off | solver

Example

Tableau:

$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓
$(\omega_0, \beta_1) \models \Box_x B$
$(\omega_1, \beta_1) \models \ell_2$ ✓
$(\omega_1, \beta_1) \models \top$ ✓
$(\omega_1, \beta_0) \models \ell_3 \to \neg B$ $(\vee)$
$(\omega_1, \beta_0) \models \ell_4 \to A$ $(\vee)$
$(\omega_1, \beta_0) \models \ell_5 \to (A \to B)$ $(\vee)$
$(\omega_1, \beta_1) \models \ell_6 \to B$ $(\vee)$

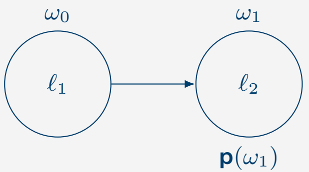
Color code:
under consideration | worked off | solver

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$
$\ell_3 \iff r \subseteq \mathbf{p}(\omega_1)$
$\ell_4 \iff s \subseteq \mathbf{p}(\omega_1)$
$\ell_5 \iff t \subseteq \mathbf{p}(\omega_1)$
$\ell_6 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$

Solver assignments:
$\mathbf{x}(\beta_1) := s$
$\mathbf{p}(\omega_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Example

Tableau:

$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓
$(\omega_0, \beta_1) \models \Box_x B$
$(\omega_1, \beta_1) \models \ell_2$ ✓
$(\omega_1, \beta_1) \models \top$ ✓
$(\omega_1, \beta_0) \models \ell_3 \to \neg B$ ✓
$(\omega_1, \beta_0) \models \ell_4 \to A$ ✓
$(\omega_1, \beta_0) \models \ell_5 \to (A \to B)$ ✓
$(\omega_1, \beta_1) \models \ell_6 \to B$ ✓
$(\omega_1, \beta_0) \models \neg \ell_3$ | $(\omega_1, \beta_0) \models \neg B$
$(\omega_1, \beta_0) \models \neg \ell_4$ | $(\omega_1, \beta_0) \models A$
$(\omega_1, \beta_0) \models \neg \ell_5$ | $(\omega_1, \beta_0) \models (A \to B)$
$(\omega_1, \beta_1) \models \neg \ell_6$ | $(\omega_1, \beta_1) \models B$

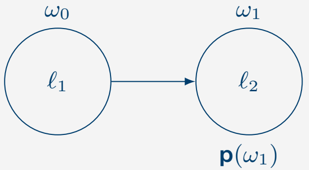
Color code:
under consideration | worked off | solver

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$
$\ell_3 \iff r \subseteq \mathbf{p}(\omega_1)$
$\ell_4 \iff s \subseteq \mathbf{p}(\omega_1)$
$\ell_5 \iff t \subseteq \mathbf{p}(\omega_1)$
$\ell_6 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$

Solver assignments:
$\mathbf{x}(\beta_1) := s$
$\mathbf{p}(\omega_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Example

Tableau:

$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓
$(\omega_0, \beta_1) \models \Box_x B$
$(\omega_1, \beta_1) \models \ell_2$ ✓
$(\omega_1, \beta_1) \models \top$ ✓
$(\omega_1, \beta_0) \models \ell_3 \to \neg B$ ✓
$(\omega_1, \beta_0) \models \ell_4 \to A$ ✓
$(\omega_1, \beta_0) \models \ell_5 \to (A \to B)$ ✓
$(\omega_1, \beta_1) \models \ell_6 \to B$ ✓
$(\omega_1, \beta_0) \models \neg \ell_3$ ✓ | $(\omega_1, \beta_0) \models \neg B$ ×
$(\omega_1, \beta_0) \models \neg \ell_4$ × | $(\omega_1, \beta_0) \models A$ ✓
$(\omega_1, \beta_0) \models \neg \ell_5$ ✓ | $(\omega_1, \beta_0) \models (A \to B)$ ×
$(\omega_1, \beta_1) \models \neg \ell_6$ × | $(\omega_1, \beta_1) \models B$ ✓

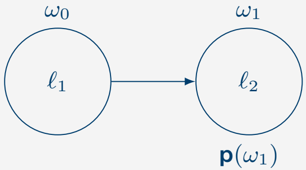
Color code:
under consideration | worked off | solver

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$
$\ell_3 \iff r \subseteq \mathbf{p}(\omega_1)$ ×
$\ell_4 \iff s \subseteq \mathbf{p}(\omega_1)$ ✓
$\ell_5 \iff t \subseteq \mathbf{p}(\omega_1)$ ×
$\ell_6 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$ ✓

Solver assignments:
$\mathbf{x}(\beta_1) := s$
$\mathbf{p}(\omega_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Example

Tableau:

$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓
$(\omega_0, \beta_1) \models \Box_x B$
$(\omega_1, \beta_1) \models \ell_2$ ✓
$(\omega_1, \beta_1) \models \top$ ✓
$(\omega_1, \beta_0) \models \ell_3 \to \neg B$ ✓
$(\omega_1, \beta_0) \models \ell_4 \to A$ ✓
$(\omega_1, \beta_0) \models \ell_5 \to (A \to B)$ ✓
$(\omega_1, \beta_1) \models \ell_6 \to B$ ✓

$(\omega_1, \beta_0) \models \neg \ell_3$ ✓	$(\omega_1, \beta_0) \models \neg B$ ×
$(\omega_1, \beta_0) \models \neg \ell_4$ ×	$(\omega_1, \beta_0) \models A$ ✓
$(\omega_1, \beta_0) \models \neg \ell_5$ ✓	$(\omega_1, \beta_0) \models (A \to B)$ ×
$(\omega_1, \beta_1) \models \neg \ell_6$ ×	$(\omega_1, \beta_1) \models B$ ✓

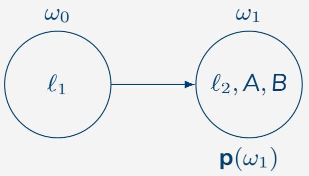
Color code:
under consideration | worked off | solver

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$
$\ell_3 \iff r \subseteq \mathbf{p}(\omega_1)$ ×
$\ell_4 \iff s \subseteq \mathbf{p}(\omega_1)$ ✓
$\ell_5 \iff t \subseteq \mathbf{p}(\omega_1)$ ×
$\ell_6 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$ ✓

Solver assignments:
$\mathbf{x}(\beta_1) := s$
$\mathbf{p}(\omega_1) := s$

Model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$



Example

Tableau:

$(\omega_0, \beta_0) \models \Box_r \neg B$
$(\omega_0, \beta_0) \models \Box_s A$
$(\omega_0, \beta_0) \models \Box_t (A \to B)$
$(\omega_0, \beta_0) \models \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$ ✓
$(\omega_0, \beta_1) \models \ell_1$ ✓
$(\omega_0, \beta_1) \models \Diamond_x \top$ ✓
$(\omega_0, \beta_1) \models \Box_x B$
$(\omega_1, \beta_1) \models \ell_2$ ✓
$(\omega_1, \beta_1) \models \top$ ✓
$(\omega_1, \beta_0) \models \ell_3 \to \neg B$ ✓
$(\omega_1, \beta_0) \models \ell_4 \to A$ ✓
$(\omega_1, \beta_0) \models \ell_5 \to (A \to B)$ ✓
$(\omega_1, \beta_1) \models \ell_6 \to B$ ✓
$(\omega_1, \beta_0) \models \neg \ell_3$ ✓ | $(\omega_1, \beta_0) \models \neg B$ ×
$(\omega_1, \beta_0) \models \neg \ell_4$ × | $(\omega_1, \beta_0) \models A$ ✓
$(\omega_1, \beta_0) \models \neg \ell_5$ ✓ | $(\omega_1, \beta_0) \models (A \to B)$ ×
$(\omega_1, \beta_1) \models \neg \ell_6$ × | $(\omega_1, \beta_1) \models B$ ✓

Color code:
under consideration | worked off | solver

Constraints:

$\ell_1 \iff \mathbf{x}(\beta_1) \subseteq r \cdot s \cdot t$
$\ell_2 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$
$\ell_3 \iff r \subseteq \mathbf{p}(\omega_1)$ ×
$\ell_4 \iff s \subseteq \mathbf{p}(\omega_1)$ ✓
$\ell_5 \iff t \subseteq \mathbf{p}(\omega_1)$ ×
$\ell_6 \iff \mathbf{x}(\beta_1) \subseteq \mathbf{p}(\omega_1)$ ✓

Solver assignments:
$\mathbf{x}(\beta_1) := s$
$\mathbf{p}(\omega_1) := s$

Eventually found a model for:
$\Box_r \neg B \wedge \Box_s A \wedge \Box_t (A \to B) \wedge \exists_{x \subseteq r \cdot s \cdot t} (\Diamond_x \top \wedge \Box_x B)$